

Tensor Part - VIII

Matrices:

A matrix of order m by n is an array of quantities A_{ij} called elements, arranged in m rows and n columns.

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}$$

It may be denoted by $[A_{ij}]$ or (A_{ij}) for $i=1, 2, \dots, m;$

$j=1, 2, \dots, n.$

* for $m=n$, matrix is square matrix of order m by m

* for $m=1 \rightarrow$ Row matrix or Row vector

* for $n=1 \rightarrow$ Column matrix or Column vector

* For the square matrix ($m=n$), the diagonal containing elements $A_{11}, A_{22}, A_{33}, \dots, A_{mm}$ is called the principal or main diagonal.

Unit matrix (I) $\rightarrow A_{ii} = 1$ and $A_{ij} = 0$ for $i \neq j$
(For square matrix only)

metric tensor:

In rectangular coordinate

the differential arc length is given by

$$ds^2 = dx^2 + dy^2 + dz^2, \quad \text{--- (1)}$$

The transformation of the above arc length to general curvilinear coordinates (discussed in earlier lecture notes) are obtain

$$ds^2 = \sum_{i=1}^3 \sum_{j=1}^3 g_{ij} du^i du^j \quad \text{--- (2)}$$

For N -dimensional space having coordinates (x^1, x^2, \dots, x^N) the line element ds given in quadratic form is called metric form or metric

$$ds^2 = \sum_{i=1}^N \sum_{j=1}^N g_{ij} dx^i dx^j$$

$$\text{or } \boxed{ds^2 = g_{ij} dx^i dx^j} \quad \text{--- (3)}$$

where we have used Einstein summation convention. (Repeated indices are summed over),

$g_{ij} \rightarrow$ may be functions of x^i with the condition that $\det(g_{ij}) \neq 0$ and the space is called as Riemannian space. For the particular case that g_{ij} are independent of $x^i \rightarrow$ space becomes Euclidean space.

Associated tensors:

Assume $A^i \rightarrow$ arbitrary contravariant vector

Now $A^i g_{ij} = A_j$

A^i and $A_j \rightarrow$ contravariant and covariant components of same vector

Again

$$A_{ij}g^{jk} = A^i g_{ij} g^{jk} = A^i \delta_i^k = A^k \quad \text{--- (4)}$$

Eq (4) shows that A_j^i & A^j_i have reciprocal relation.

Tensors A^i_j and A_j^i are called associate tensors.

Moment of inertia tensor:

The angular momentum $\vec{L} = \sum_i m_i (\vec{r}_i \times \vec{v}_i)$ --- (5)

$m_i \rightarrow$ mass of the i th point mass
 $\vec{r}_i \rightarrow$ position vector of m_i w.r.t. origin
 $\vec{v}_i \rightarrow$ linear velocity of m_i

If $\vec{\omega} \rightarrow$ angular velocity vector of the body

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

$$\therefore \vec{L} = \sum_i m_i [\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)] = \sum_i m_i [\omega r_i^2 - \vec{r}_i (\vec{r}_i \cdot \vec{\omega})]$$

we have used $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ --- (6)

Now writing in cartesian components

$$\vec{L} = (L_x, L_y, L_z), \quad \vec{r}_i = (x_i, y_i, z_i)$$

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

Now from eq (6)

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

$$L_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z$$

$$L_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z$$

$$\text{--- (7)}$$

where, $I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$, $I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$, $I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i, \quad I_{yz} = I_{zy} = -\sum_i m_i y_i z_i$$

$$I_{zx} = I_{xz} = -\sum_i m_i z_i x_i$$

Eq (7) shows that \vec{L} is not parallel to $\vec{\omega}$. We write

$$\boxed{L_i = I_{ij} \omega_j} \text{ --- (8), } \omega_j \rightarrow \text{vectors}$$

$I_{ij} \rightarrow$ tensor (from quotient law it is clear from eq 8)